NOTE ON FORCED VIBRATION OF A THIN NON-HOMOGENEOUS CIRCULAR PLATE WITH A CENTRAL HOLE

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(*Received July* 26, 1967; *Resubmitted January* 30, 1968).

ABSTSACT. In this paper we have considered the vibration produced in a thin non-homogeneous circular plate with a central hole by an application of a periodic force acting on the internal boundary. Young's modulus and density are supposed to vary linearly with the radius vector.

We have determined the displacement produced due to the elastic vibrations produced in a thin circular non-homogeneous plate by an application of an internal pressure which varies with time. Here Young's modulus E and density ρ are taken as $E = E_0 r$ and $\rho = \rho_0 r$, where r is the radius vector and E_0 and ρ_0 are constants. The stress distribution is chosen symmetrical with respect to the axis through the centre of the plate and perpendicular to the plane (xy-plane) of it. By symmetry it follows that shearing stress $\sigma_{r\theta}$ vanishes.

The equation of motion is

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \frac{\partial^2 u}{\partial t^2} \qquad \ldots \quad (1)$$

The stress strain relation are

$$\left.\begin{aligned} E.e_{rr} &= \sigma_{rr} - \nu\sigma_{\theta\theta} \\ E.e_{\theta\theta} &= \sigma_{\theta\theta} - \nu\sigma_{rr} \end{aligned}\right\} \qquad \ldots \quad (2)$$

The strain components are

$$e_{rr} = \frac{\partial u}{\partial r} \text{ and } e_{\theta\theta} = \frac{u}{r} \qquad (3)$$

From equations (2) and (3) we have

$$\left.\begin{aligned} \sigma_{rr} &= \frac{E}{1-\nu^2}\left[\frac{\partial u}{\partial r} + \nu\frac{u}{r}\right] = \frac{E_0}{1-\nu^2}\left[r\frac{\partial u}{\partial r} + \nu u\right] \\ \sigma_{\theta\theta} &= \frac{E}{1-\nu^2}\left[\frac{u}{r} + \nu\frac{\partial u}{\partial r}\right] = \frac{E_0}{1-\nu^2}\left[u + \nu r\frac{\partial u}{\partial r}\right] \end{aligned}\right\} \qquad \ldots \quad (4)$$

Since we have $E = E_0 r$ and $\rho = \rho_0 r$ ... (5)

From equations (1) and (4) we have

$$r^2 \frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r} - (1-\nu)u = \frac{\rho_0}{E_0}(1-\nu^2)\, r^2 \frac{\partial^2 u}{\partial t^2} \quad \ldots \quad (6)$$

Boundary conditions:

$$\left.\begin{array}{lll} \sigma_{rr} = -P(1-\cos wt) & r = r_0, & t > 0 \\ \sigma_{rr} = 0 & r = r_1, & t > 0 \\ \text{The initial conditions are that,} & & \\ \text{at } t = 0, \quad u = \dfrac{\partial u}{\partial t} = 0, & r_0 \leqslant r \leqslant r_1 & \end{array}\right\} \quad \ldots \quad (7)$$

Multiplying both sides of the equation (6) by e^{-pt} and integrating with respect to t from 0 to ∞, we obtain the ordinary differential equation

$$r^2 \frac{d^2\bar{u}}{dr^2} + 2r\frac{d\bar{u}}{dr} + [-(1-\nu) - a^2 r^2 p^2]\bar{u} = 0 \quad \ldots \quad (8)$$

where $a^2 = \frac{\rho_0}{E_0}(1-\nu^2)$ and $\bar{u} = \int_0^\infty u e^{-pt} dt.$

Solving equation (8) we have

$$\bar{u} = r^{-\frac{1}{2}}[AI_k(\text{apr}) + BK_k(\text{apr})] \quad \ldots \quad (9)$$

where $k = \frac{\sqrt{5-4\nu}}{2}$

Taking the value of the Poisson's ratio $\nu = 0.25$Eq.uation (9) becomes

$$\bar{u} = r^{-\frac{1}{2}}[AI_1 \text{ par}) + BK_1(\text{par})] \quad \ldots \quad (10)$$

where $a^2 = \frac{\rho_0}{E_0} \cdot \frac{15}{16}$

Taking Laplace transform of equation (7) we have

$$\left.\begin{array}{ll} \bar{\sigma}_{rr} = \dfrac{E_0}{1-\nu^2}\left[r\dfrac{\partial \bar{u}}{\partial r} + \nu\bar{u} \right] = \dfrac{-Pw^2}{p(p^2+\omega^2)} & r = r_0 \\ \bar{\sigma}_{rr} = \dfrac{E_0}{1-\nu^2}\left[r\dfrac{\partial \bar{u}}{\partial r} + \nu\bar{u} \right] = 0 & r = r_1 \end{array}\right\} \quad \ldots \quad (11)$$

Substituting equation (10) in Eq. (11) we have

$$\left.\begin{aligned} &A\left[\, I_1(apr_0)-\frac{4}{5}r_0 apI_0(apr_0)\,\right] \\ &\qquad +B\left[\, K_1(apr_0)+\frac{4}{5}ar_0pK_0(apr_0)\right] = \frac{4}{5}r_0^{\frac{1}{2}}\frac{P\omega^2}{p(p^2+\omega^2)} \\ &A\left[I_1(apr_1)-\frac{4}{5}ar_1pI_0(apr_1)\right]+B\left[K_1(apr_1)+\frac{4}{5}ar_1pK_0(apr_1)\right]=0 \end{aligned}\right\} \quad \dots \quad (12)$$

If we write the equation (12) in the form

$$AL_0+BM_0-Q = 0$$
$$AL_1+BM_1+0 = 0$$

where

$$L_{0,1} = I_1(apr_{0,1})-\frac{4}{5}r_{0,1}apI_0(apr_1)$$

$$M_{0,1} = K_1(apr_{0,1})+\frac{4}{5}r_{0,1}apK_0(apr_{0,1}) \quad \dots \quad (13)$$

and

$$Q = \frac{4}{5}r_0^{\frac{1}{2}}\frac{P\omega^2}{p(p^2+\omega^2)}$$

Xolving for A and B we have

$$A = \frac{M_1Q}{L_0M_1-M_0L_1} \quad \text{and} \quad B = \frac{-L_1Q}{L_0M_1-M_0L_1} \quad \dots \quad (14)$$

Byf(13) and (14), equation (10 becomes

$$\bar{u} = r^{-\frac{1}{2}}.\,Q.\,\frac{[M_1I_1(arp)-L_1K_1(arp)]}{L_0M_1-M_0L_1}$$

$$= r^{-\frac{1}{2}}.\,\frac{4}{5}r_0^{\frac{1}{2}}.\,\frac{P\,.\,w^2}{p(p^2+w^2)}.\,\frac{F(p)}{G(p)} \quad \dots \quad (15)$$

where

$$F(p) = \left[\, K_1(apr_1)+\frac{4}{5}ar_1pK_0(apr_1)\right] I_1(arp)$$

$$-\left[\, I_1(apr_1)-\frac{4}{5}apr_1I_0(apr_1)\,\right] K_1(arp)$$

and

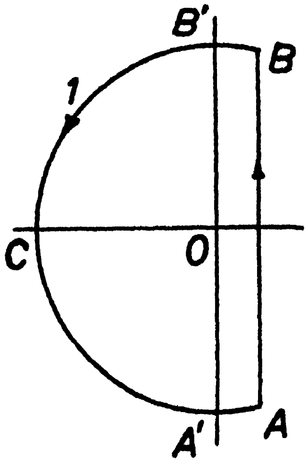
$$G(p) = \left[I_1(apr_0) - \frac{4}{5} r_0 ap I_0(apr_0) \right] \left[K_1(apr_1 + \frac{4}{5} ar_1 p K_0(apr_1) \right]$$

$$- \left[K_1(apr_0) + \frac{4}{5} ar_0 p K_0(apr_0) \right] \left[I_1(apr_1) - \frac{4}{5} apr_1 I_0(apr_1) \right]$$

Applying Laplace's Inversion Theorem we have for the displacement

$$\frac{5}{4} \frac{r_0^{-\frac{1}{2}} r^{\frac{1}{2}}}{Pw^2} u = \int_{c-i\infty}^{c+i\infty} \frac{F(p)}{p(p^2+w^2)G(p)} e^{pt} dp \qquad \dots \quad (16)$$

To evaluate the integral on the right-hand side of the above equation, consider the integral taken round the closed contour consisting of a line at a distance c from the imaginary axis and the portion (lying to the left) of a circle whose centre is the origin and whose raidus is $R = \dfrac{(n+\frac{1}{2})\pi}{(r_1 - r_0)a}$ — chosen so that the contour avoids all poles of the integrand. It can be shown that the limit of the integral around this circular arc tends to zero as n tends to infinity so that Canchy's theorem enables us to replace, the line integral in equation (16) by the sum of the residues of the function $\dfrac{p(p^2+w^2)G(p)}{F(p)e^{pt}}$ lying to the left of the line $R(p) = c$



The contour used in the evaluation of the integral.
(16)

The poles of this are at the points $p = 0$, $p = \pm iw$ and at the roots of the transcendental equation $G(p) = 0$, which have been shown by Tranter (1942) to be simple and purely imaginary. They will be written in the form $p = \pm i\alpha_s$.

If $w \neq \alpha_s$ for any value of s, the sum of the residues of the function at $p = \pm iw$ is easily seen to be.

$$\frac{-F(iw)\cos wt}{w^2 G(iw)} \text{ and that at } p=0 \text{ is } \frac{F(0)}{w^2 G(0)} \qquad \dots \quad (17)$$

The sum of the residues of the integrand at the remaining poles is

$$\sum_{s=1}^{\infty} \frac{1}{w^2-\alpha_s^2}\left[\frac{e^{i\alpha_s t}F(i\alpha_s)}{\left(\alpha\frac{dG}{d\alpha}\right)_{\alpha=\alpha_s}} + \frac{e^{-i\alpha_s t}F(-i\alpha_s)}{\left(\alpha\frac{dG}{d\alpha}\right)_{\alpha=-\alpha_s}}\right] \qquad \dots \quad (18)$$

After little reduction we can show that

$$\left(\alpha\frac{dG}{d\alpha}\right)_{\alpha=\alpha_s} = \left(\alpha\frac{dG}{d\alpha}\right)_{\alpha=-\alpha_s}$$

$$= \xi\,(\gamma^2\alpha_s^2 r_0^2 a^2 - 2\gamma+1) - \frac{1}{\xi}\,(\gamma^2\alpha_s^2 r_1^2 a^2 - 2\gamma+1)$$

where $\gamma = \frac{4}{5}$ and $\xi = \dfrac{J_1(\alpha_s r_1 a) - (\gamma\alpha_s r_1 a)J_0(\alpha_s r_1 a)}{J_1(\alpha_s r_0 a) - (\gamma\alpha_s r_0 a)J_0(\alpha_s r_0 a)}$

$$= \frac{j(r_1)}{j(r_0)} \qquad \text{say}$$

and that

$$F(i\alpha_s) = F(-i\alpha_s) = -\tfrac{1}{2}.\,\pi\{[Y_1(\alpha_s r_1 a) - \gamma\alpha_s r_1 a Y_0(\alpha_s r_1 a)]J_1(\alpha_s r a) - j(r_1)Y_1(\alpha_s r a)\}$$

Substituting these results in equation (17) and (18) we obtain finally

$$\frac{5}{4}\frac{u}{Pr_0^{\frac{1}{2}}} = r^{-\frac{1}{2}}\left[\frac{r_0}{r}\left\{\frac{r^2+(2\gamma-1)r_1^2}{(2\gamma-1)(r_1^2-r_0)}\right\} - \frac{F(iw)}{G(iw)}\cos wt \right.$$
$$\left. + 2w^2\sum_{s=1}^{\infty}\frac{F(i\alpha_s)(w^2-\alpha_s^2)^{-1}j(r_0)j(r_1)\cos(\alpha_s t)}{(\gamma^2\alpha_s^2 r_0^2 a^2-2\gamma+1)j(r_1) - \gamma^2\alpha_s^2 r_1^2 a^2 - 2\gamma+1)j(r_0)}\right] \qquad \dots \quad (19)$$

Here summation is taken over all positive roots of the equation $G(i\alpha_s) = 0$. From equation (19) stresses can be calculated.

My sincere thanks are due to Dr. A. K. Das for suggesting the problem.

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